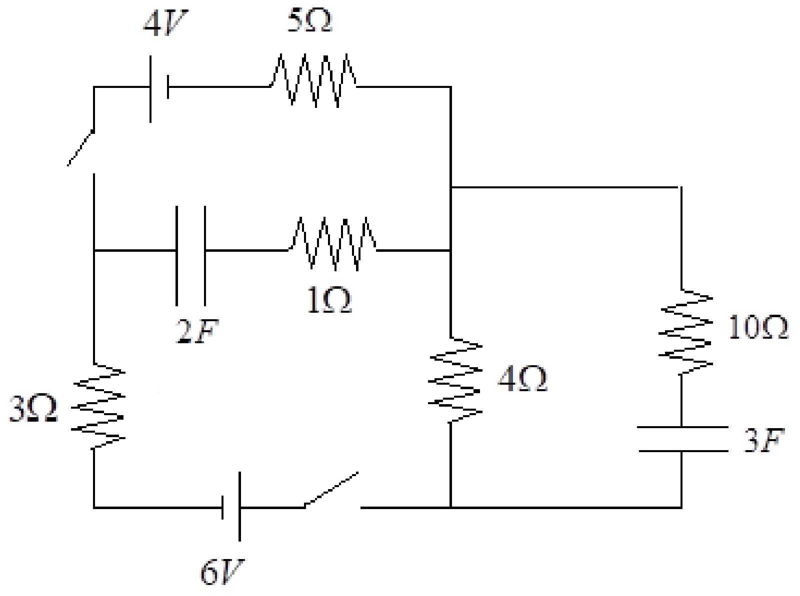
# B.4 RC Circuits: Charging

Many circuits combine capacitors and resistors together, especially the aforementioned circuits that are used to charge a bank of capacitors, in order to subsequently release the high voltage charge across a motor, or other device.  So with that in mind, let's consider a typical setup, and relevant general principles.




**t = 0**: when flip the switch the capacitors will as yet be uncharged and there will be an initial current through their (and all) wires serving to charge them.

**0 < t < ∞**: as time progresses, the capacitors will gradually (or quickly) charge, and the current through their wires will attenuate.

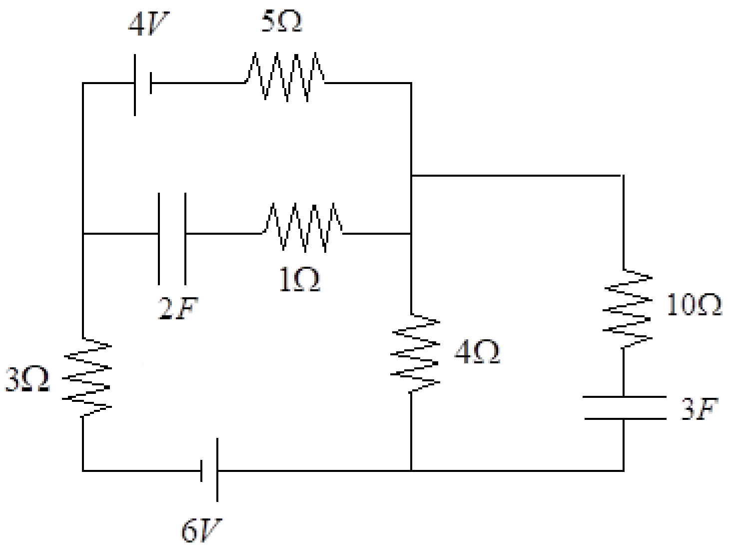
**t = ∞**: Eventually, the capacitors will be fully charged and the current through *their* wires will cease.

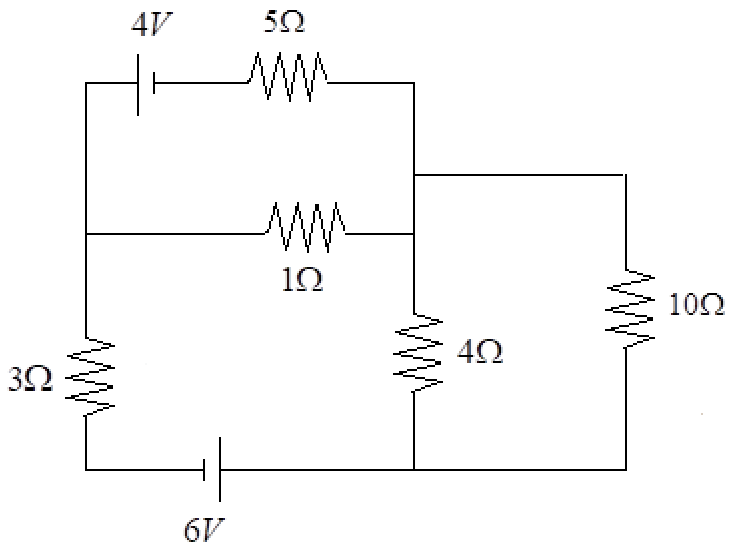
For now we will focus on analyzing the t = 0 and t = ∞ situations, to determine the initial and final currents/charges in the circuit.

An analysis of the 0 < t < ∞ window requires differential equations, and due to the mathematical complexity, we'll tackle only the simplest relevant scenarios.

# B.4 RC Circuits: Charging

t = 0: What are the charges on all capacitors, and currents in all wires?



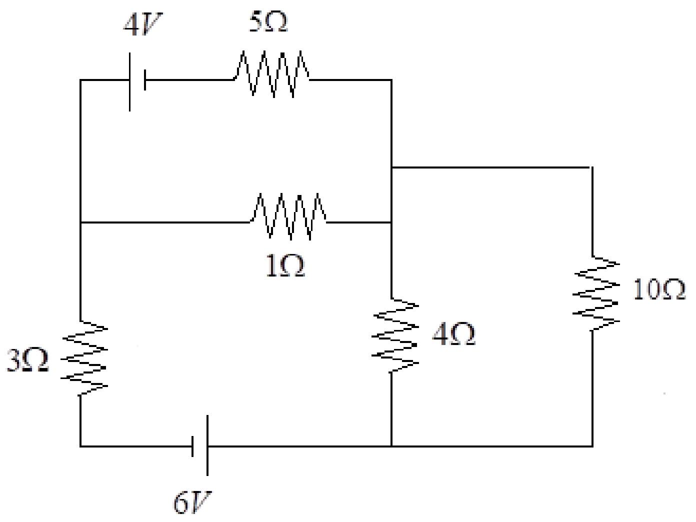



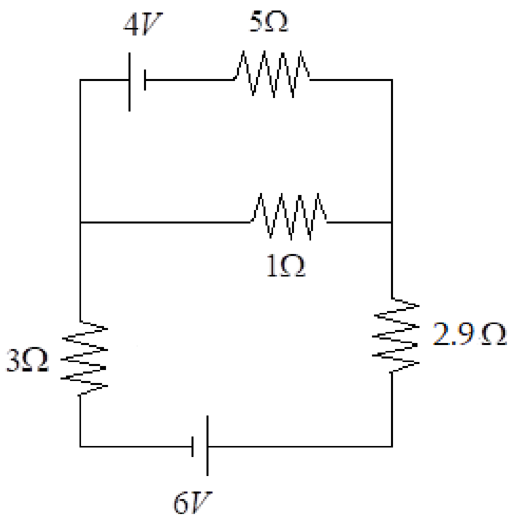

As stated, initially, all charges are zero. So there is no potential difference across them (since ΔV = q/C). Therefore they offer no initial resistance to current and we can pretend they're not even there. In other words, we can pretend they're shorted.

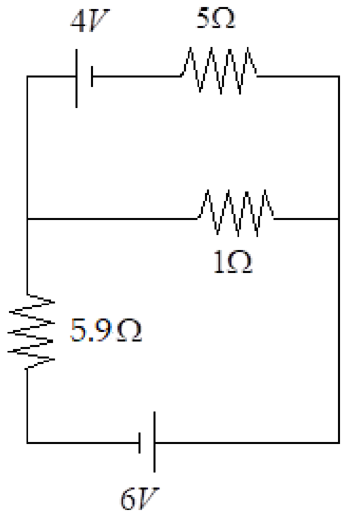
At this point we can use Kirchoff's laws and/or equivalent resistance stuff, as necessary, to get the currents.

# B.4 RC Circuits: Charging

t = 0: What are the charges on all capacitors, and currents in all wires?




Let's combine the 4Ω and 10Ω resistors in parallel.

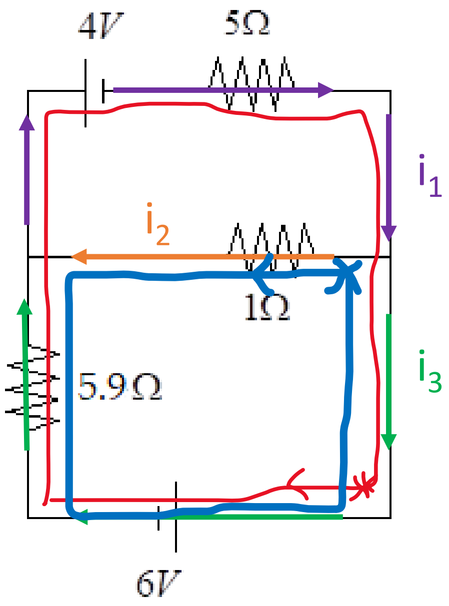
$$R_{parallel} = (4^{-1} + 10^{-1})^{-1} = 2.9\Omega$$

Now let's combine the 3Ω and 2.9Ω resistor in series

$$R_{series} = 3 + 2.9 = 5.9\Omega$$

# B.4 RC Circuits: Charging

t = 0: What are the charges on all capacitors, and currents in all wires?




Now we'll use Kirchoff's laws,

KCL: $-i_1 + i_2 + i_3 = 0$

KVL (red): $-6 - 5.9i_3 - 4 - 5i_1 = 0$

$$5i_1 + 5.9i_3 = -10$$

KVL (blue): $-1i_2 + 5.9i_3 + 6 = 0$

$$i_2 - 5.9i_3 = 6$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 5 & 0 & 5.9 \\ 0 & 1 & -5.9 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 6 \end{pmatrix}$$

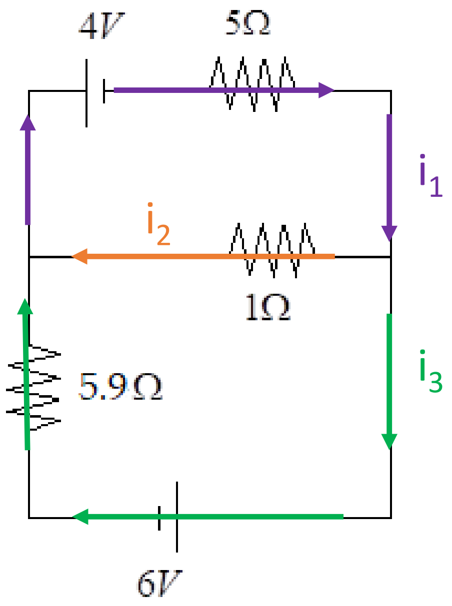
$$\longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 5 & 0 & 5.9 \\ 0 & 1 & -5.9 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -10 \\ 6 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -0.83 \\ 0.16 \\ -0.99 \end{pmatrix}$$

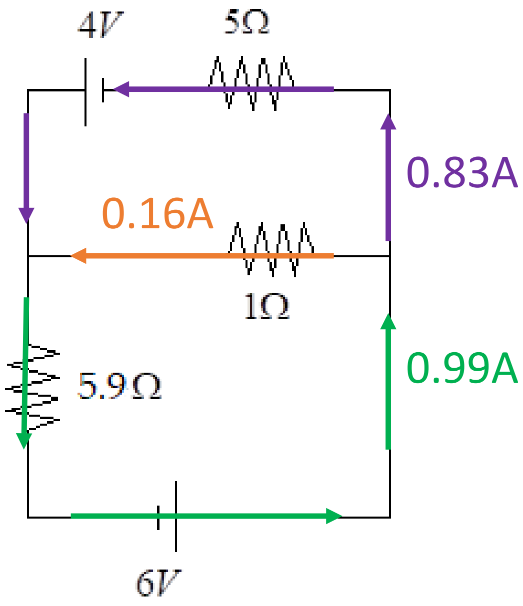
Now matrix-ize it, or whatever you prefer.

# B.4 RC Circuits: Charging

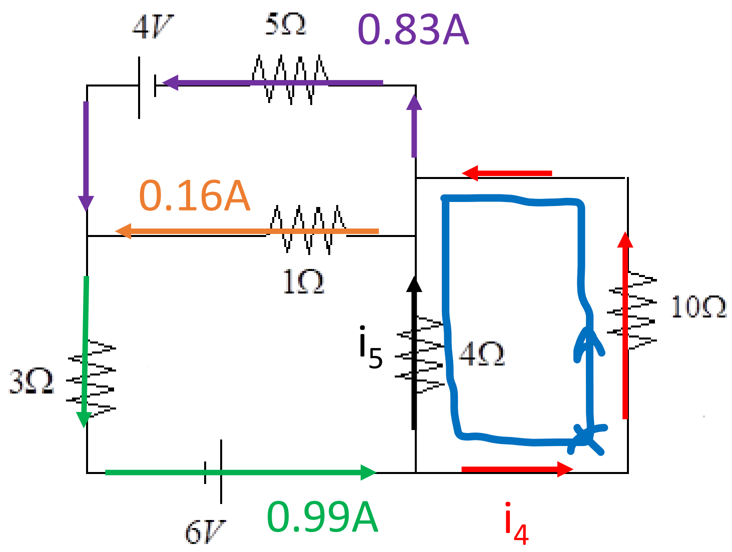
t = 0: What are the charges on all capacitors, and currents in all wires?




$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -0.83 \\ 0.16 \\ -0.99 \end{pmatrix}$$




The negative signs indicate current is flipped in those wires. So we have circuit above:

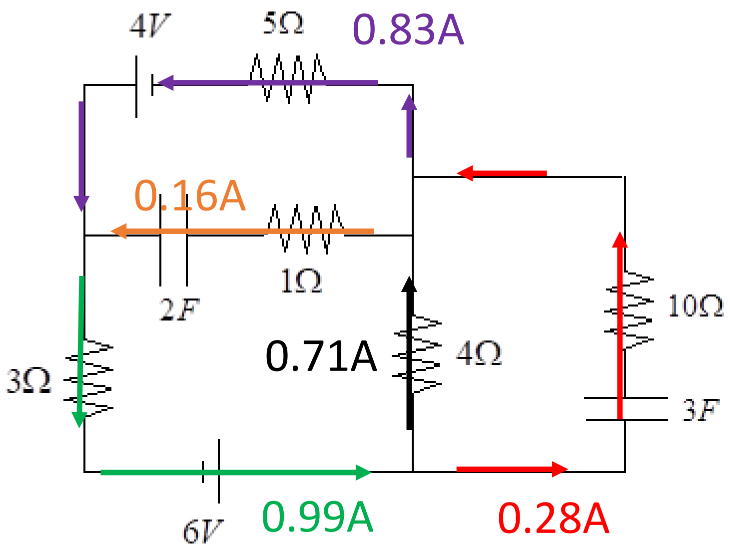



Going back to the original circuit, we have to resolve the current in that branch we collapsed in parallel.

$$\left.\begin{aligned} \text{KCL:}\quad & i_4 + i_5 = 0.99 \\ \text{KVL:}\quad & -10 i_4 + 4 i_5 = 0 \end{aligned}\right\} \begin{pmatrix} i_4 \\ i_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -10 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0.99 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.28 \\ 0.71 \end{pmatrix}$$

# B.4 RC Circuits: Charging

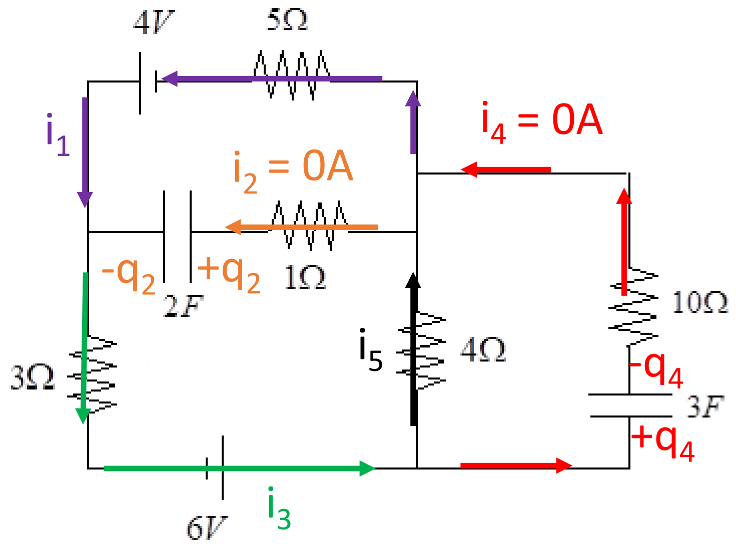
t = 0: What are the charges on all capacitors, and currents in all wires?




So this is our result, finally, well, initially:
All charges on capacitors are zero.
Currents are as given.

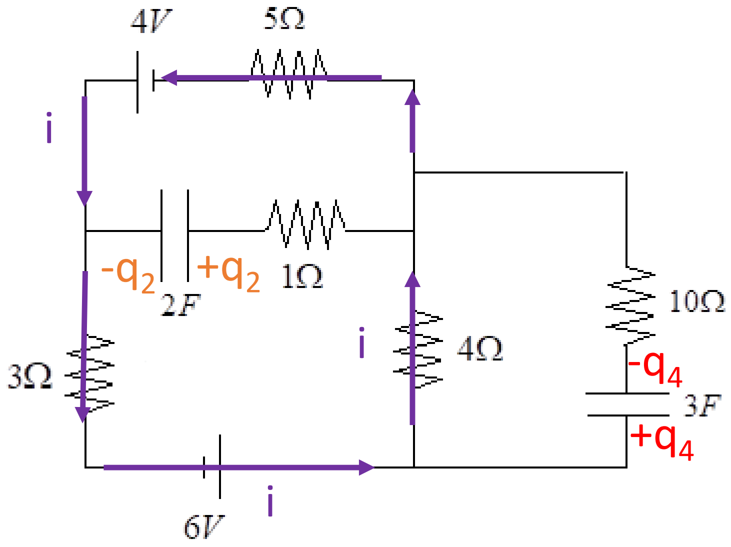
# B.4 RC Circuits: Charging

t = ∞: What are the charges on all capacitors, and currents in all wires?



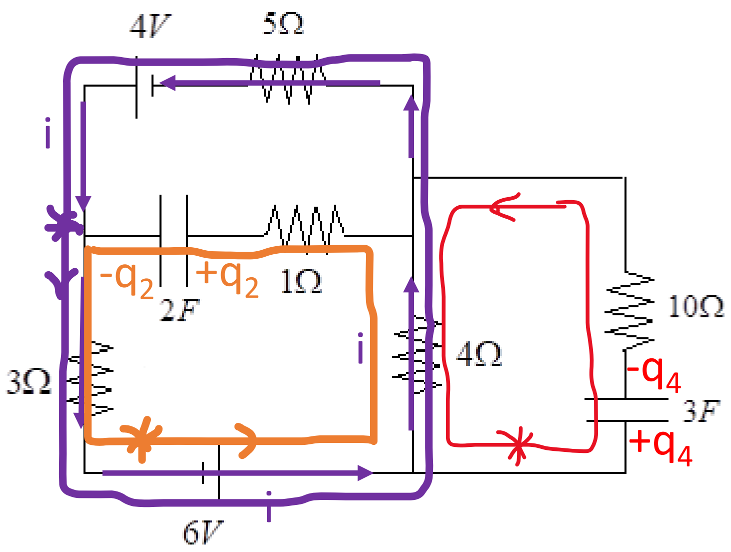

Once the capacitors have charged, the currents in their wires will be zero.

At this point, we should observe that $i_1$ will become $i_3$, will become $i_5$. They are all the same. So we could just write the circuit like this:



# B.4 RC Circuits: Charging

t = ∞: What are the charges on all capacitors, and currents in all wires?




To get i, we can do KVL around the purple current loop.

$$-3i + 6 - 4i - 5i + 4 = 0$$

$$12i = 10 \longrightarrow i = 0.83\,\mathrm{A}$$

To get $q_2$ we can do KVL around the orange loop.

$$+6 - 4i - 1\cdot 0 - \frac{q_2}{2} - 3i = 0$$

$$6 - 4(0.83) - \frac{q_2}{2} - 3(0.83) = 0$$

$$q_2 = 2\cdot\left[6 - 7(0.83)\right] = 0.33\mathrm{C}$$

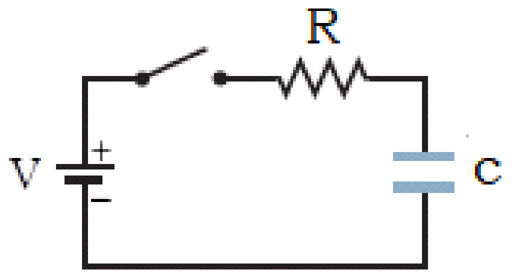
To get $q_4$ we can KVL the red loop.

$$-\frac{q_4}{3} - 10\cdot 0 + 4i = 0$$

$$q_4 = 12i = 12(0.83) = 10\mathrm{C}$$

# B.4 RC Circuits: Charging

Now we'll analyze what happens for in between t's. But we'll have to examine a simpler setup to do so in any detail. The most general circuit we can 'simply' analyze is one which can be reduced, via equivalent capacitance or resistance techniques, to the following form. We'll aim to get the charge on the capacitor as a function of time. But first…



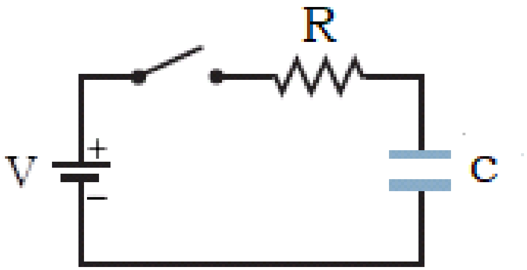

t = 0: What is the charge on the capacitor right after we flip the switch? What is the current?

$$q_0 = 0 \qquad i_0 = V / R$$

t = ∞: What is the charge on the capacitor a long time after we flip the switch? What is the current?

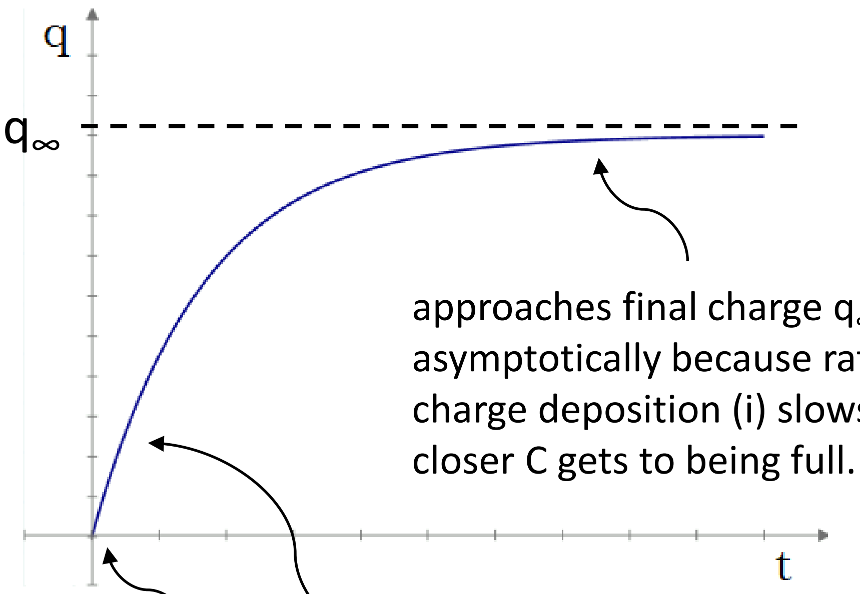
$$q_\infty = CV \qquad i_\infty = 0$$

# B.4 RC Circuits: Charging




0 < t < ∞: What is the charge on the capacitor at intermediate times?

And what do we expect the q(t) curve to look like?




approaches final charge $q_\infty = CV$ asymptotically because rate of charge deposition (i) slows, the closer C gets to being full.

starts at zero of course

initial slope is initial rate of charge deposition (i), $i_0 = V/R$.

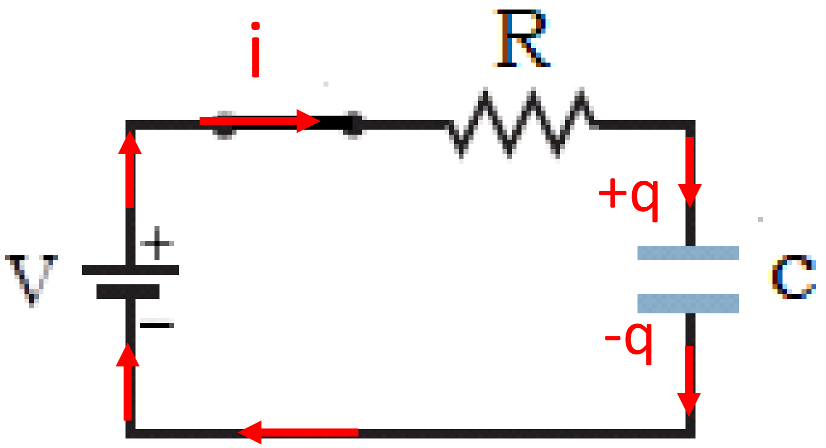
How do we expect C, R, and V to affect the charging time?

Increasing R ⟶ longer time because slower current delivers charge less rapidly?

Increasing C ⟶ longer time because more charge to fill?
special case: plain wire has zero capacitance
and would take no time to 'fill'
so maybe time is indeed ~ C

Increasing V ⟶ shorter time because faster current?
Or longer because greater final charge?
Maybe cancels out?

# B.4 RC Circuits: Charging




0 < t < ∞: What is the charge on the capacitor at intermediate times?

Well time to get quantitative. Current is drawn, and it will charge the capacitor as it flows. Applying KVL, in the same direction as the current (just 'cause) we get:

$$+V - iR - \frac{q}{C} = 0$$

$$V - \frac{dq}{dt}R - \frac{q}{C} = 0$$

$$\frac{dq}{dt} = \frac{CV - q}{RC}$$

$$\frac{dq}{q - CV} = -\frac{1}{RC}dt$$

$$\int_0^q \frac{dq}{q - CV} = -\int_0^t \frac{dt}{RC}$$

$$\ln|q - CV|\Big|_0^q = -\frac{t}{RC}$$

$$\ln\left|\frac{q - CV}{-CV}\right| = -\frac{t}{RC}$$

$$\frac{q - CV}{-CV} = e^{-\frac{t}{RC}}$$

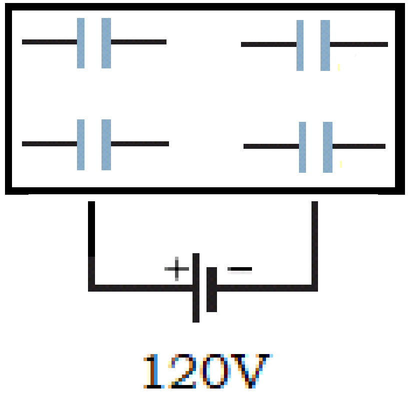
$$q = CV(1 - e^{-\frac{t}{RC}})$$

*τ = time constant*: roughly time it takes to charge 63% full. Conforms with our expectations that larger R and C slow charging process. Apparently V has no effect.

$$q(t) = q_\infty(1 - e^{-t/\tau}) \qquad \tau = RC$$

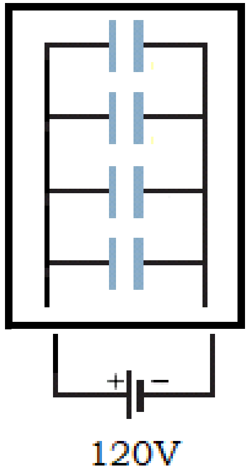
# B.4 RC Circuits: Charging

Defibrillators don't quite work this way, but they *could*. Say a defibrillator consists of four 20mF capacitors. And say we hook it up to a 120V battery.



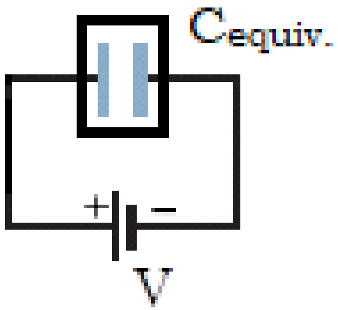

How should the capacitors be connected to each other to maximize the amount of charge they acquire, when connected to the battery?

Parallel




What charge will the capacitors store?

This is the charge on the equivalent capacitor…




$$C_{eq.} = C + C + C + C = 4C$$
$$= 4(20\,\mathrm{mF}) = 80\,\mathrm{mF}$$

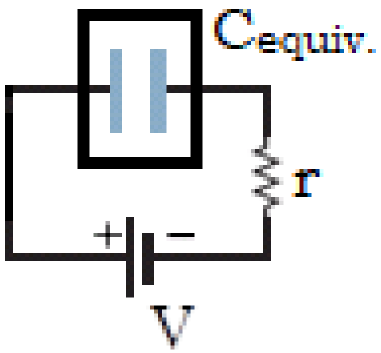
$$Q = C\Delta V$$
$$= (80\mathrm{mF})(120\mathrm{V})$$
$$= 9.6\,\mathrm{C}$$

The defibrillator will charge faster if its connected to a newer battery. Why?

because the older a battery gets, the more its terminals tend to corrode when exposed to air. This increases the resistance of the battery's terminals, and therefore the resistance in the circuit ($\tau$ = RC)

# B.4 RC Circuits: Charging

Say the battery has an internal resistance r = 2Ω.
What would be the charge on the capacitors after 0.10s?
What would be the current through the battery at that time?




Well,

$$q(t) = q_\infty(1 - e^{-t/\tau})$$

$$\tau = RC = (2\Omega)(80\text{mF}) = 160\,\text{ms}$$

$$q_\infty = 9.6\,\text{C}$$

Therefore,

$$q(t) = 9.6(1 - e^{-t/0.160})$$

And,

$$q(0.10\text{s}) = 9.6(1 - e^{-0.10/0.160}) = 4.5\,\text{C}$$

$$i(0.10\text{s}) = \left.\frac{dq}{dt}\right|_{t=0.10\text{s}} = \left.\frac{9.6}{0.160}e^{-t/0.160}\right|_{t=0.10\text{s}} = 32\,\text{A}$$

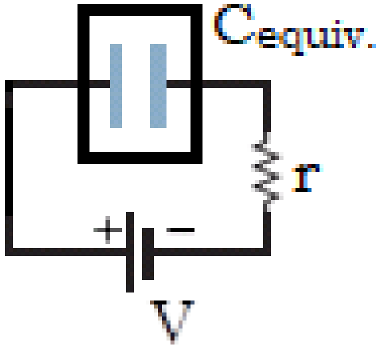
What would be the potential difference across the capacitor at this time, and what energy will it have stored?

We can get all of these things with q:

$$\Delta V_{cap} = \frac{q}{C} = \frac{4.5\,\text{C}}{80\,\text{mF}} = 56\,\text{V}$$

$$PE_{cap} = \frac{1}{2}C\Delta V_{cap}^2 = \frac{1}{2}(0.080\text{F})(56\text{V}) = 125\,\text{J}$$

# B.4 RC Circuits: Charging


How long will it take for the capacitors to charge to 90% capacity?

This'll be:

$$q = q_{\infty}(1 - e^{-t/\tau})$$

$$0.90(9.6) = 9.6(1 - e^{-t/0.160})$$

$$0.90 = 1 - e^{-t/0.160}$$

$$e^{-t/0.160} = 0.10$$

$$\ln(e^{-t/0.160}) = \ln(0.10)$$

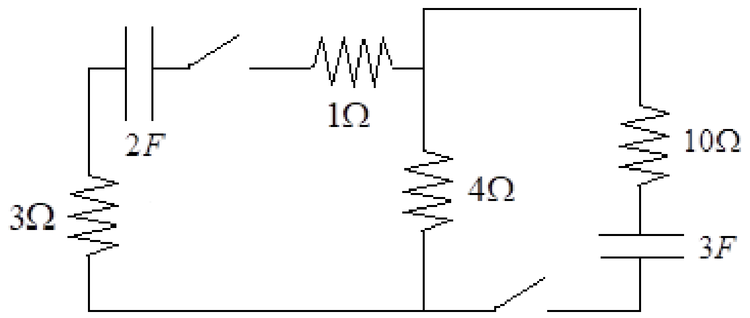
$$-\frac{t}{0.160} = \ln(0.10)$$

$$t = -(0.160)\ln(0.10)$$

$$t = 0.37\text{s}$$

# B.4 RC Circuits: Discharging

After the capacitors have been all charged up, we typically discharge them over some device.
So let's consider what will happen when a network of capacitors discharges




**t = 0**: the capacitors are initially charged, and as soon as the switch is flipped (closed), they will begin discharging with some initial current through their (and all) wires.

**0 < t < ∞**: as time progresses, the capacitors will gradually (or quickly) discharge, and the current through their (and all, assuming no external power source) wires will attenuate.

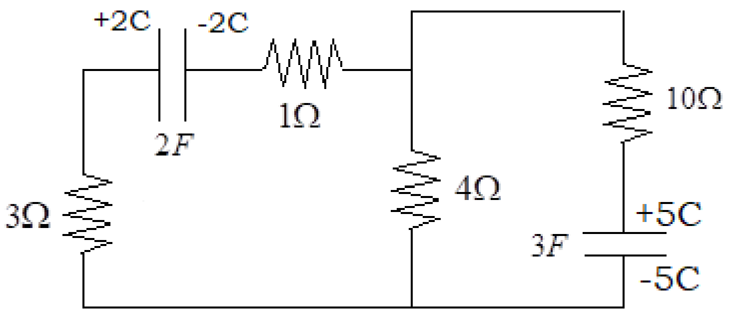
**t = ∞**: Eventually, the capacitors will be fully discharged and the current through their (and all, assuming no external power source) wires will cease.

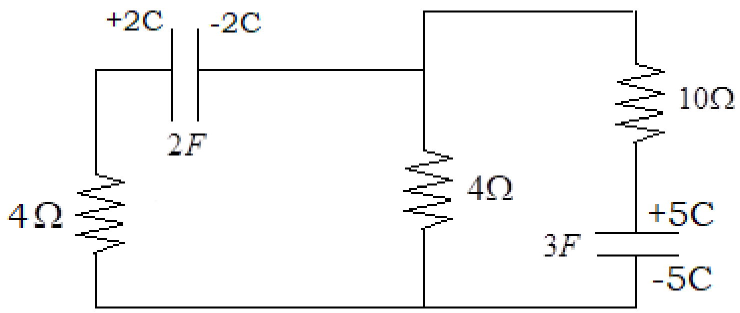
For now we will focus on analyzing the t = 0 and t = ∞ situations, to determine the initial and final currents/charges in the circuit.

An analysis of the 0 < t < ∞ window requires differential equations, and due to the mathematical complexity, we'll tackle only the simplest relevant scenarios.

# B.4 RC Circuits: Discharging

t = 0: Assuming the initial charges shown, what will be the initial currents in the circuit?



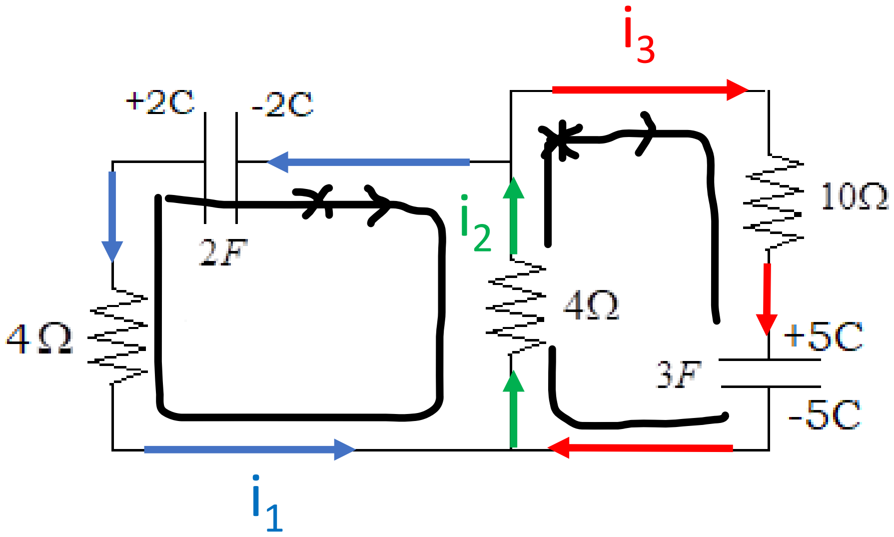




We could first combine the two left branch resistors in series. So I will.

At this point we have to use Kirchoff's laws to get the currents.

# B.4 RC Circuits: Discharging

t = 0: Assuming the initial charges shown, what will be the initial currents in the circuit?




Label currents:

Do Kirchoff's laws:

KCL: $-i_1 + i_2 - i_3 = 0$

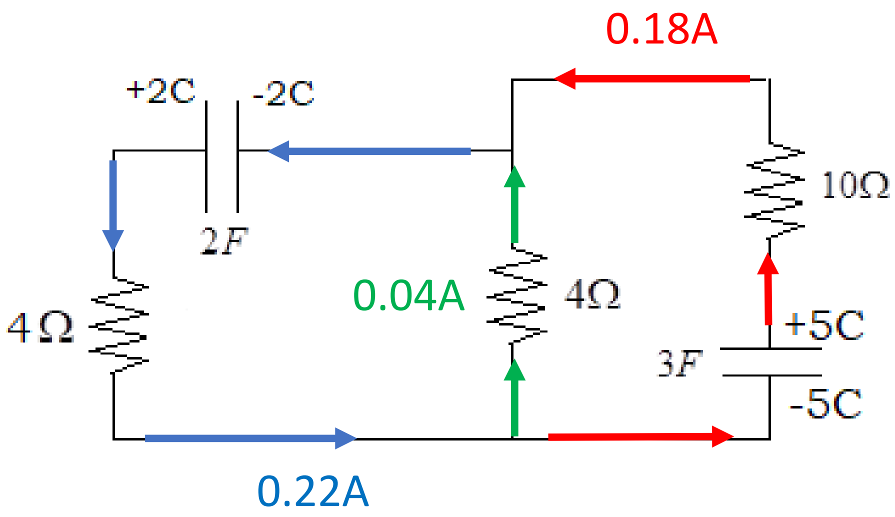
KVL (left): $+4i_2 + 4i_1 - \frac{2}{2} = 0 \longrightarrow 4i_1 + 4i_2 = 1$

KVL (right): $-10i_3 - \frac{5}{3} - 4i_2 = 0 \longrightarrow 4i_2 + 10i_3 = -\frac{5}{3}$

Solve:

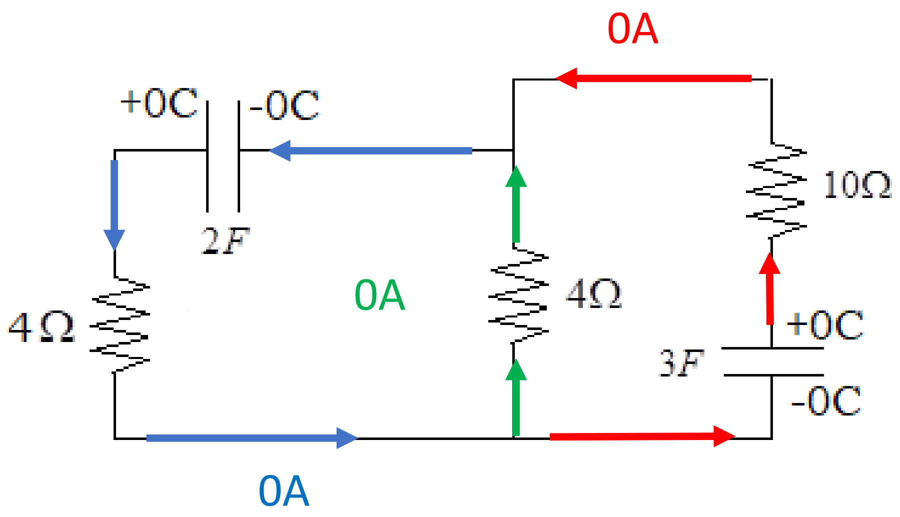
$$\begin{pmatrix} -1 & 1 & -1 \\ 4 & 4 & 0 \\ 0 & 4 & 10 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -5/3 \end{pmatrix} \longrightarrow \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 4 & 4 & 0 \\ 0 & 4 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \\ -5/3 \end{pmatrix} = \begin{pmatrix} 0.22 \\ 0.04 \\ -0.18 \end{pmatrix}$$

So circuit looks like this, initially:



# B.4 RC Circuits: Discharging

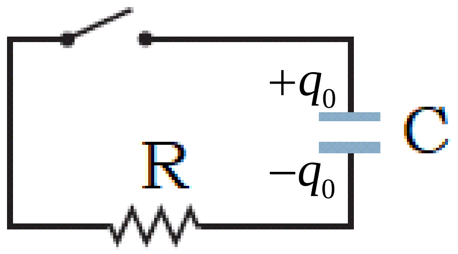
t = ∞: What are the final charges on all capacitors, and currents in all wires?




All capacitors will have discharged and all currents will have decayed to zero.

# B.4 RC Circuits: Discharging

Now we'll analyze what happens for in between t's.  Again, we'll have to examine a simpler setup to do so in any detail. The most general circuit we can 'simply' analyze is one which can be reduced, via equivalent capacitance or resistance techniques, to the following form. We'll aim to get the charge on the capacitor as a function of time.  But first…



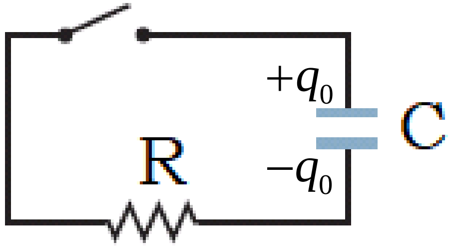

t = 0: What is the charge on the capacitor right after we flip the switch?  What is the current?

$$q_0 = q_0 \qquad i_0 = \frac{q_0 / C}{R}$$

t = ∞: What is the charge on the capacitor a long time after we flip the switch?  What is the current?

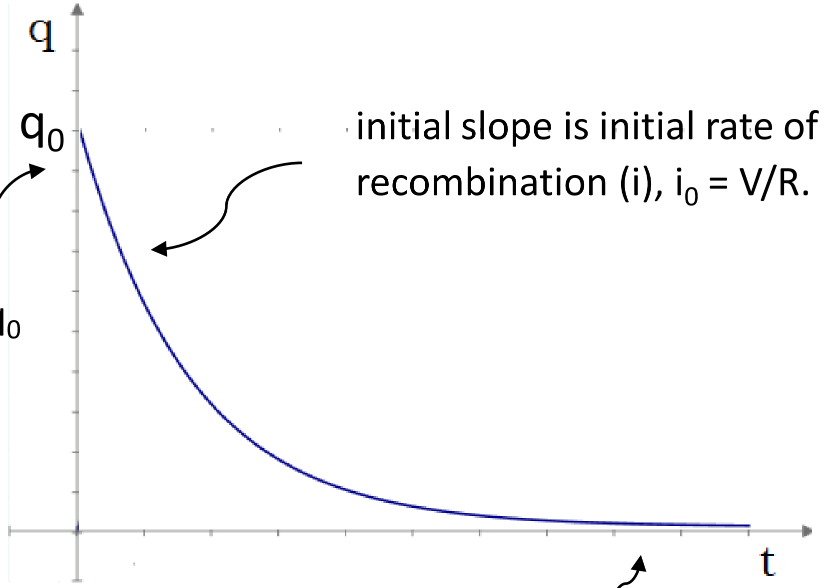
$$q_\infty = 0 \qquad i_\infty = 0$$

# B.4 RC Circuits: Discharging




0 < t < ∞: What is the charge on the capacitor at intermediate times?

And what do we expect the q(t) curve to look like?




initial slope is initial rate of charge recombination (i), $i_0$ = V/R.

starts at $q_0$ of course

approaches final charge $q_\infty = 0$ asymptotically because rate of charge recombination (i) slows, the closer C gets to being empty.

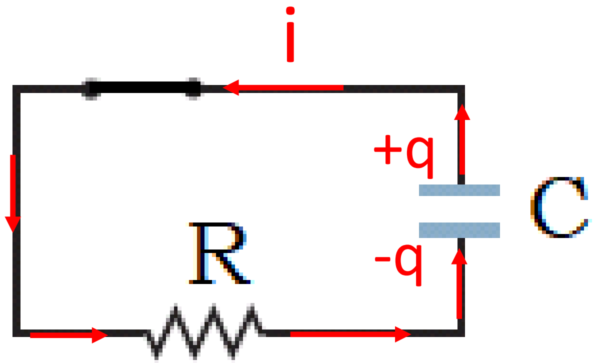
How do we expect C, and R to affect the charging time?

Increasing R ⟶ longer time because slower current recombines charge less rapidly?

Increasing C ⟶ longer time because more charge to recombine:

Is decay discharging time constant the same as the charging time constant?

# B.4 RC Circuits: Discharging




0 < t < ∞: What is the charge on the capacitor at intermediate times?

Well time to get quantitative. Current is drawn, and it will discharge the capacitor as it flows. Applying KVL, in the same direction as the current we get:

$$-iR + \frac{q}{C} = 0$$

$$\frac{dq}{dt}R + \frac{q}{C} = 0$$

$$\frac{dq}{dt} = \frac{-q}{RC}$$

$$\frac{dq}{q} = -\frac{1}{RC}dt$$

$$\int_{q_0}^{q} \frac{dq}{q} = -\int_{0}^{t} \frac{dt}{RC}$$

$$\ln|q|\Big|_{q_0}^{q} = -\frac{t}{RC}$$

$$\ln\left|\frac{q}{q_0}\right| = -\frac{t}{RC}$$

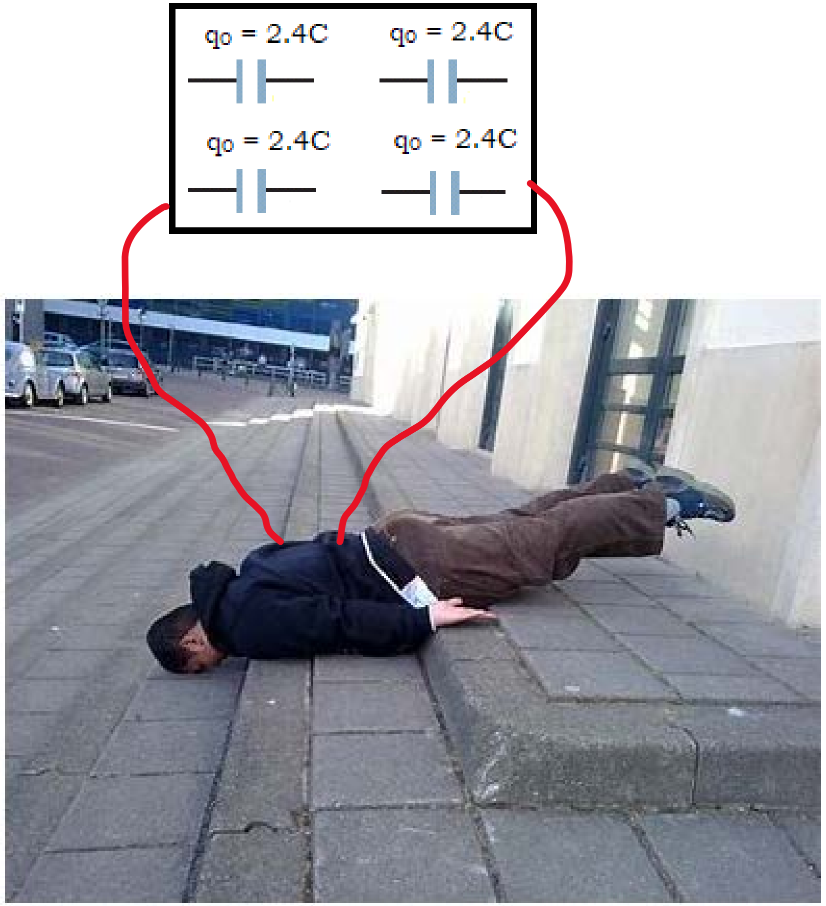
$$\frac{q}{q_0} = e^{-\frac{t}{RC}}$$

τ = *(same) time constant*: roughly time it takes to discharge 63%.

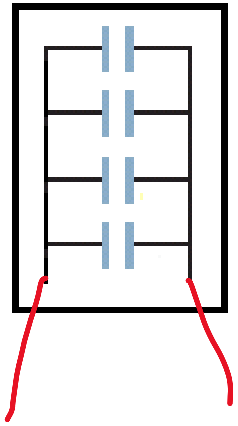
$$q(t) = q_0 e^{-t/\tau} \qquad \tau = RC$$

# B.4 RC Circuits: Discharging

Going back to the defibrillator….we had charged each 20mF capacitor to 120V, resulting in a total stored charge of 9.6C, or 2.4C per capacitor. Now we're going to discharge them over this guys chest.



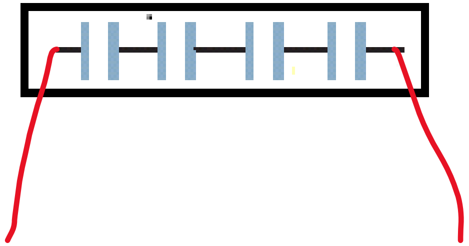

How should the capacitors be connected to each other to maximize the amount of charge that's delivered to his chest? What would this be. What energy would be delivered?



We'd want to do parallel. That way, all 9.6C will be delivered.

$$PE_{total} = \frac{1}{2}C_{eq.}\Delta V^2 = \frac{1}{2}(80\text{mF})(120\text{V})^2 = 576\,\text{J}$$

How should the capacitors be connected to each other to produce the largest current through his chest? What energy would be delivered?



We'd want series, so the potential difference across his chest would be 4 × 120V = 480V (larger ΔV results in larger current). But observe only 2.4C would be delivered to his chest. Energy is….

$$PE_{total} = \frac{1}{2}C_{eq.}\Delta V^2 = \frac{1}{2}(5\text{mF})(480\text{V})^2 = 576\,\text{J}$$

# B.4 RC Circuits: Discharging

So they both deliver the same energy, but which connection would deliver the greatest power?

That would be series, because it has the smallest equivalent capacitance, and hence the smallest time constant $\tau = RC_{eq}$.

So say we hook it up this way and discharge the defibrillator. A human chest has a resistance of about 67Ω. How much charge will have been delivered after 0.5s? What max current would be delivered. How long until 90% of the energy was delivered.

$$q(t) = q_0 e^{-t/\tau} \qquad q_0 = 2.4\,\mathrm{C} \qquad \tau = RC = (67\Omega)(0.005\mathrm{F}) = 0.33\mathrm{s}$$

$$q(0.5\mathrm{s}) = (2.4)e^{-0.5/0.33} = 0.53\mathrm{C} \longrightarrow$$ 1.87C has been delivered.

Max current is initial current: $$i_{\max} = \frac{\Delta V}{R} = \frac{480\mathrm{V}}{67\Omega} = 7.2\mathrm{A}$$

Energy *remaining* is: $$PE_{total} = \frac{q^2}{2C} = \frac{(q_0 e^{-t/\tau})^2}{2C} = \frac{q_0^2 e^{-2t/\tau}}{2C} = \frac{(2.4)^2 e^{-2t/0.33}}{2(0.005)} = 576e^{-6.1t}$$

So want to solve: $$PE_{total} = (0.10)576 \longrightarrow 576e^{-6.1t} = 0.10(576)$$

$$\longrightarrow -6.1t = \ln(0.10)$$

$$\longrightarrow t = 0.37\mathrm{s}$$